21st Century Relativity: A Euclidean Isomorphism via the Gudermannian and λ 6-Group

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Abstract

This paper presents a novel isomorphism of special relativity using hyperbolic trigonometry and the Gudermannian function to unify spacetime symmetries. Lorentz transformations are reinterpreted as hyperbolic rotations, with the Gudermannian explaining the cosmic speed limit and momentum projections. A 6-group derived from Möbius transforms reveals group-theoretic invariants. The framework preserves experimental predictions but rejects non-Euclidean geometry, deriving the energy-momentum dispersion relation from Pythagorean principles. This approach highlights the physicality of celerity and challenges the privileging of Minkowski spacetime.

Keywords

special relativity, Gudermannian function, λ 6-group, hyperbolic trigonometry, celerity, Lorentz transformation, eigenspace, Möbius transformation

1 Introduction

Before presenting the main premise of this paper, some preliminary issues merit discussion. There is a pervasive attitude that only individuals with degrees in physics are qualified—and thus eligible—to submit papers for peer review. The rationalization is that others are unlikely to contribute anything new or relevant. The author disagrees. Even among physics graduates, half ranked in the lower half of their class. A Bachelor of Science graduating at the top of the class should not be disqualified merely because physics was not the chosen specialization.

While there may have once been justification for screening frivolous papers, the era of artificial intelligence offers a solution: eliminate human pre-screening for new submissions, along with eligibility restrictions. Beyond qualifications, there is bias in topics deemed worthy of consideration, regardless of the author. Relativity is often hailed as the most tested theory, with criticism dismissed as misunderstanding. Coupled with demands for falsifiability, this seems compelling—but it is not logical. Falsifiability does not prove a theory; it only establishes minimum worthiness for debate. Failure rejects a theory, as it should, but passing is merely a baseline, not confirmation.

In mathematics, some theories are inherently unfalsifiable yet correct, such as isomorphisms. By definition, no experiment distinguishes one version from another. This attitude grants preferential treatment to established theories over competitors predicting identical measurements. The differences between isomorphisms do not affect experimental outcomes. Experimental confirmation alone does not prove any theory. As Einstein remarked, hundreds of experiments could not prove him right, but one could prove him wrong.

Minkowski's spacetime is such an isomorphism—it predicts the same outcomes as Einstein's but requires extreme assumptions. While spacetime is one facet, Minkowski overlooked its multi-dimensional nature. Even in his geometry, space and time span a plane at sub-light speeds, not mere projections. Spacetime must have at least two axes, with space and time as composites. This blending occurs at all velocities, even in stationary frames.

A more fundamental coordinate system exists where axes remain perpendicular despite relative velocity, without Lorentz blending. The clue is the invariance of axis direction with velocity—an eigenvector property. Hermann Bondi termed it k-calculus, with "k" as an eigenvalue. The Lorentz transformation in eigenspace coordinates differs from Minkowski's. The Minkowski matrix is fully populated (for parallel velocity), while the eigenspace matrix is diagonal, with no off-diagonal blending.

Eigenvectors' dot product is zero, despite being space-time composites. Like Minkowski's light-cone eigenvectors, they are "orthogonal" but not dot-product zero. The cross-product of eigenvectors is well-defined, with magnitude equal to Minkowski's metric—the true invariant.

The Lorentz transformation is hyperbolic trigonometry. The invariant $s^2 = (ct + r)(ct - r) = c^2t^2 - r^2$ is hyperbolic magnitude. The spacetime interval is the coordinate transformation between Minkowski and hyperbolic coordinates. The hyperbolic angle is the boost, independent of velocity since velocity is a hyperbolic rotation.

This paper presents a different isomorphism predicting identical measurements but preserving Euclidean geometry and the Pythagorean identity. It rejects the inconsistent notion that the hypotenuse is always invariant, central to Minkowski's metric. Conventional circular trigonometry follows this, but hyperbolic trigonometry does not. Graphically, a tangent to any radius vector intersects the horizontal axis, forming a triangle with base 1 (radius), altitude tangent, hypotenuse secant. Minkowski made the base the new hypotenuse, distorting the Pythagorean identity for his invariant. The base is the hypotenuse of the smaller triangle. In hyperbolic trigonometry, the invariant is the base, not the hypotenuse. Applying the standard Pythagorean identity yields hypotenuse^2 - altitude^2 = base^2, consistent with cosine projections.

Minkowski corrupted geometry for internal consistency, but it was unnecessary. This paper shows a framework preserving the same predictions but without rejecting Euclidean geometry. All new theories must, of course, account for all the same behavior as the old theory. But to be a credible replacement, it is usually expected that some new property that can be tested experimentally will be included. That is not exactly possible, here, because this is an isomorphism. As already explained, it predicts exactly the same results as the theory it competes with. And it cannot be tested by any new experiment, either. That being said, this theory offers something different. It provides mathematical proofs for properties that relativity merely asserts based on empirical observation. Chief among these is the 2nd Postulate, the cornerstone of relativity, followed by a rational explanation for relativistic momentum that does not involve the deprecated concept of relativistic mass, which was never valid. In his "Little Lecture Series", he demonstrated that mass is a relativistic invariant of the Lorentz transformation of 4-momentum, and cannot vary with velocity. But the most insightful comment he made about it was that "perhaps momentum is not proportional to velocity." It is, however, directly proportional to celerity, the supposedly fictional velocity. Mass and momentum are both physical properties. It makes no sense that their ratio is not physical as well.

Books will be written about the implications of the tilt angle, but this paper does not claim to address every possible application. But, a sufficient number will be explained in enough detail to warrant serious further research. We begin at the beginning, as did Einstein in his first paper, "On the Electrodynamics..."

2 On measurement

Before even discussing relativity, Einstein explained that it was necessary for observers to agree as to how they measured phenomena so that they could talk about events with confidence. He maintained that a theoretically ideal measuring system would give all observers a consistent way to describe intervals of time and distance. His theoretical system was not very practical in most cases, so he allowed that any alternative, such as trigonometry, would be acceptable, as long as it would produce the same results as the ideal system would, if it were possible. His procedure for measuring length involved a system of rigid measuring rods (like surveyors') placed end-to-end along the axis of measurement. Easy to describe, but hard to implement over large distances or with relatively moving objects. But the idea of an ideal is that no physical system can do any better.

He then applies his ideal system, hypothetically, to an object moving at a substantial fraction of lightspeed, by applying a Lorentz transformation to the coordinates of the endpoints of the interval being measured. He is forced to report that his ideal system could not measure 100 per cent of the rest length of the object. Under the rules of logic, this failure was a contradiction that should have killed the premise. Instead, he invoked the ad hoc corrections of time dilation and length contraction, so he could now say that his method measured these effects correctly. He would additionally assert that these effects were necessarily physical in order to justify the observed invariance of lightspeed. Not all of his contemporaries agreed with him, but the success of the rest of special relativity at explaining a wide variety of observations eventually won out.

The idea was not without problems, however. Although Einstein asserted that the effect had to be physical, there were nagging contradictions associated with that. In the first place, this was before general relativity, and there was no awareness of the crushing amount of gravity required to overcome the internuclear forces. Objects simply do not shrink on their own, especially in the absence of extreme gravity. Einstein's theory required objects to shrink just because of relative velocity. Worse, it didn't matter if the object were moving and the observer standing still, or the observer were moving and the object was standing still. That's relativity for you. Worse still, if more than 1 observer moving at different velocities attempted to measure the same stationary object at the same time, both observers would get different measurements. This was "explained" by saying that since all observers got exactly the correct measurement for the relative velocity of their own frame, that there was no contradiction. The fact remains that even if you accept that "logic", the contradictory measurements could not be physical in nature. After all, if they were illusions, then they could be different without a real contradiction. But if the effect were not physical, the measurements of invariant lightspeed were, and this was a more serious contradiction.

So, let's go back to the beginning. The protocol which Einstein asserted is literally an illustration of the dot product of two vectors with zero included angle between them. In the isomorphism under consideration, velocity is accounted for by a Lorentz transformation that is a pure, hyperbolic rotation. Every possible boost angle has a unique gudermannian, tilt, angle, so every velocity corresponds to a tilt angle between the moving and the stationary frames. Tilt angle is relative, just like velocity, and the dot product is commutative and even, so if tilt is the included angle, it makes no difference which frame is defined to be stationary. While this tilt angle is not visible, we don't have to see it to use it in the dot product. Any vector that is rotated away from a reference unit can only project the cosine fraction of its magnitude, and the process is symmetrical.

In other words, suppose we have two frames that are synchronized and relatively stationary. An event that occurs at (ct,r) has the same coordinates in both frames. But if one frame is set into motion, the event occurs at (ct',r'). According to the Lorentz transformation, the time interval is defined by ct = ct' and the distance by r = r', under conditions of coincidence of clock locations and simultaneity of distance measurements. Of course, with relatively moving frames, it is non-trivial to decide that clock locations are actually coincident, and we are all aware of the relativity of simultaneity. In our geometry, both time and distance rotate away from the reference plane, as if they are embedded in the walls of a cone which closes like an umbrella as velocity increases. This has the interesting consequence that the two perpendicular axes begin to approach each other. In the limit of a 90 degree tilt away from the plane, they degenerate into a normal vector and are indistinguishable.

As they rotate away from the plane, they project the cosine fraction of themselves back onto the plane. This is the only part of the length of the vector that the stationary observer can measure. So ct' = ct cos(tilt) and r' = r cos(tilt). But tilt is defined by relative velocity as tilt = Arcsin(v/c), or sin(tilt) = v/c. If we substitute this expression into the empirical definition of the Lorentz factor, we find that = sec(tilt) and 1/ = cos(tilt). Then ct' = ct/ and r' = r/. These are absolutely equivalent to ct = ct' and r = r', time dilation and length contraction as defined by Einstein, but without any reference to clock location or simultaneity. It turns out that the stipulation of the dot product protocol supersedes the individual stipulations of coincidence and simultaneity. In addition, it requires that the magnitude of the moving interval remains invariant in its own frame of reference. Nothing shrinks, but we can measure the illusion. There is no longer even a hint of contradiction, because every observer is measuring a 3D shadow of the 4D magnitude.

By replacing the false assertion that it is possible to measure 100 per cent of any moving interval with an ideal measuring system with a dot product protocol, we no longer expect that a moving measurement will be the same as a stationary one. The geometry says we should expect a cosine projection. The physical measurements confirm that's exactly what we get. This is the mathematical basis for the claim that all observers get exactly the correct measurements for the relative velocity of their frame of reference. All of this presupposes that the two frames were once synchronized. Two travelers who meet in deep space cannot decide which one is moving faster, nor can they tell which one is more time dilated. But relativity can do no better. This is not a legitimate objection to the tilt angle theory.

There is a long list of measured properties which follow the dot product measurement protocol. Here is a list of the most common: contracted length is the cosine projection of Proper length; Proper time is the cosine projection of coordinate time; measured velocity is the cosine projection of celerity; Newtonian momentum is the cosine projection of relativistic momentum; and, at the top of the list, a small increment of tilt angle is the cosine projection of a small increment of boost, d(tilt) = cos(tilt) d(boost), a rearrangement of the differential equation that defines the gudermannian, d(boost)/d(tilt) = .

3 Invariance of lightspeed

There have been attempts to prove the 2nd Postulate in the past. None have been successful, and the ones that seemed to be have all been revealed to include the assumption of invariance somewhere in the proof. Circular logic never proved anything. Suppose we want to solve the differential equation. It is not in the correct form for solution. If we move the cosine factor to the LHS, we have to integrate the secant function, a non-trivial integral. The alternative is replacing it with a function of boost, but that requires knowing the solution. However, it is not necessary to actually solve the differential equation to identify the hyperbolic function of boost that is identical to the cos(tilt), when tilt is the gudermannian of boost. It is based on fundamental geometry and does not depend on physics.

The solution is based on the unit circle and the unit hyperbola. Since both have two perpendicular axes of symmetry, we can confine our attention to the 1st quadrant with no loss of generality. That way, all of our coordinates off axis are positive definite. We start with two arbitrary points. That's a total of 4 arbitrary coordinates. One of them is a point on the unit circle, (x,y), and the other is a point on the unit hyperbola, (w,z). These two conditions impose to constraints on what coordinates we are allowed to select. They can be expressed as $x^2+y^2 = 1$ and $w^2-z^2 = 1$. Since w is never 0, we can rewrite the second condition as $1-(z/w)^2 = (1/w)^2$, or $(1/w)^2+(z/w)^2 = 1$. Given 4 arbitrary variables, we can impose another constraint. This time, let x = 1/w. This implies that $x^2+y^2 = x^2+(z/w)^2 = 1$. More to the point, y = z/w, with no ambiguity, because all coordinates are positive. This gives us 6 basic relationships, which turn out to be the elements of a λ 6-group. They are: w = 1/x w/z = 1/y 1/z = x/y 1/w = x z/w = y z = y/x We can now convert to polar and hyperbolic coordinates, using x = cos(tilt), y = sin(tilt), w = cosh(boost) and z = sinh(boost). Our 4th stipulation is that tilt = gd(boost). Assuming the 4 conditions are not incompatible, the 6 identities that result are: cosh(boost) = sec(tilt) coth(boost) = csc(tilt) csch(boost) = cot(tilt) sech(boost) = cos(tilt) tanh(boost) = sin(tilt) sinh(boost) = tan(tilt) Are these valid? If we implicitly differentiate any one of them with respect to either angle, the result is the same in all 12 cases, d(boost)/d(tilt) = sec(tilt) = cosh(boost). Without integrating, we have solved the differential equation, and shown that it is solved by a 6-group. as a test, we now have an identity for cos(tilt) = sech(boost). If we multiply both sides of the equation by an integrating factor, cos(tilt), and apply the identity, the result is cos(tilt) d(tilt) = sech^2(boost) d(boost) d(sin(tilt)) = d(tanh(boost)). If we integrate this, we get sin(tilt) = tanh(boost), along with the boundary condition, gd(0) = 0. None of the derivation has anything to do with physics or experiments. Now we can proceed to examining the differential equation itself.

Since gd(0) = 0, and v = c sin(tilt) = c tanh(boost), velocity near 0 tilt is also near 0. This means the differential equation can be approximated for very small velocities by d(tilt) d(boost). This is the domain of Newtonian physics As a result of the approxiation, tilt sin(tilt) tan(tilt) boost tanh(boost) sinh(boost) v/c u/c. This is why Newton's formula for momentum is incorrect. It's only a low speed approximation where velocity celerity. Momentum is invariant mass * celerity, but at low speeds it is easily mistaken for mass * velocity. At higher speeds, velocity and celerity diverge, and Newtonian momentum diverges from relativistic momentum. For mid-range boosts, we have the identity already solved, v = c sin(tilt) = c tanh(boost), with tilt = gd(boost). Near light speed, it gets interesting.

Since boost composes linearly, as uniform increments of boost are added, the total boost grows linearly and without limit. As it approaches infinity, the tanh(boost) asymptotically approaches 1, as must the sin(tilt). The Arcsin(1) is $\pi/2$, and at that angle, v approaches c and the cos(tilt) approaches 0. In other words, the contribution ofthe uniform increment of boost towards the tilt angle becomes vanishingly small. At the limit, the cosine is 0. None of any increment of boost can add to the tilt, and as long as the tilt doesn't change, neither does its cosine. The differential equation can be approximated in this limit by d(tilt) = 0 or tilt = constant, which is $\pi/2$. This is a hard limit. Boost, along with γ, has approached infinity, v has approached c, and momentum has approached infinity, along with celerity, γv. There is no boost greater than infinity, so there is no measured velocity greater than c.

Faster than light is literally impossible. This cannot be deduced from the empirical formula for the Lorentz factor, which appears to only be singular at c. But the analytical definition is more restrictive. There simply is no velocity greater than c. Just goes to show that it is never legitimate to extend an empirical formula beyond the range of the data it was created from. Since u = c sinh(boost) = c tan(tilt) and v = c tanh(boost) = c sin(tilt), we can now define the maximum observable speed as the unique limit of the cosine projections of celerity as boost approaches infinity. Apparently, the universe has no problem with infinite celerity, just infinite momentum. All of the counter-intuitive properties of lightspeed turn out to be the universe following laws of mathematics as regards infinity. You can't add a finite increment to infinity, nor can you multiply it by a scalar. It is the same for all observers, so the limit of its cosine projections is also the same for all observers, the 2nd Postulate.

4 The λ 6-Group and Gudermannian Function

The formal λ 6-group arises from Möbius transformations relating trigonometric and hyperbolic functions, closed under composition [4]. The Cayley table is isomorphic to modulo 6 arithmetic. The 6-group governs the identities, defined above from considerations of geometry, and mapped here to physical properties represented by their common Greek symbols:

$$\gamma = \cosh\eta = \sec\theta \tag{1}$$
$$\beta = \tanh\eta = \sin\theta \tag{2}$$
$$\beta\gamma = \sinh\eta = \tan\theta \tag{3}$$
$$\frac{1}{\beta} = \coth\eta = \csc\theta \tag{4}$$
$$\frac{1}{\beta\gamma} = \operatorname{csch}\eta = \cot\theta \tag{5}$$
$$\frac{1}{\gamma} = \operatorname{sech}\eta = \cos\theta \tag{6}$$

The **Gudermannian function**, denoted $\operatorname{gd}(\eta)$, maps the boost η to the tilt angle θ:

$$\theta = \operatorname{gd}(\eta) = \arctan(\sinh\eta) = \arcsin(\tanh\eta) \tag{7}$$

Of special interest is the half-angle identity:

$$\tan\left(\frac{\theta}{2}\right) = \tanh\left(\frac{\eta}{2}\right) \tag{8}$$

Note that the identities make the integral form of the gudermannian function obvious. The Bondi eigenvalue, k, is e^{η}, and it equals $\sec(\theta)+\tan(\theta)$. This is easily confirmed by the identity, $e^{\eta} = \cosh(\eta)+\sinh(\eta)$.

5 Euclidean Derivation of the Dispersion Relation

The dispersion relation $E^2 = p^2c^2+m^2c^4$ emerges from Euclidean right triangles and the Pythagorean identity. Consider a right triangle with legs mc^2 (rest energy) and pc (momentum energy), hypotenuse E:

$$E^2 = (mc^2)^2 + (pc)^2 \tag{9}$$

6 Critique of Minkowski Spacetime

The figure illustrates the derivation of the dispersion relationship from Euclidean geometry. The first panel is pure geometry, with a superposition of dimension lines from 3 different coordinate systems, Cartesian, Polar and Hyperbolic. It is a basic ruler and compass construction, except for the hyperbola. But even without the hyperbola, the rest of the construction is ruer and compass, and the intersection of the cross-hairs from the endpoints of the larger arc always locate a point which is on the unit hyperbola.

The other panels simply scale the geometric drawing by relativistic invariant quantities. The last two panels show the difference between momentum, a vector quantity, and energy, a scalar. The dispersion relationship is more correctly about momentum than energy, because it is an application of the Pythagorean Identity to two perpendicular vectors. The correct drawing for energy is the last panel, where it is treated as a scalar.

The velocity panel shows that velocity in spacetime is the hypotenuse of the triangle formed by velocity in time and celerity in space. The cosine projection of this velocity is the unit circle, consistent with the dot product protocol for measurement. So, it isn't that the velocity in spacetime is constant, only that the measurable velocity in spacetime is constant. In any case no one has ever actually measured it. It's hypothetical and it's wrong. It is inconsistent with Minkowski's assertion that time is just a projection of spacetime. If lightspeed is invariant in space, then it should also be invariant in time. The Lorentz transformation attributes all of time dilation to the duration of the interval, while requiring that the velocity during the interval remains constant. Minkowski's model assumes constant spacetime velocity c, but hyperbolic projections show velocity exceeding c in spacetime (point P in Figure 1). The bi-valued invariant $x^2 - y^2 =$ constant limits its modeling of projections.

7 Relativistic momentum and invariant mass

8 Conclusion and Future Work

This isomorphism unifies relativity through the Gudermannian and 6-group, preserving predictions while restoring Euclidean geometry. Future work explores Maxwell's equations via sequency decomposition and variable scalar fields [5].

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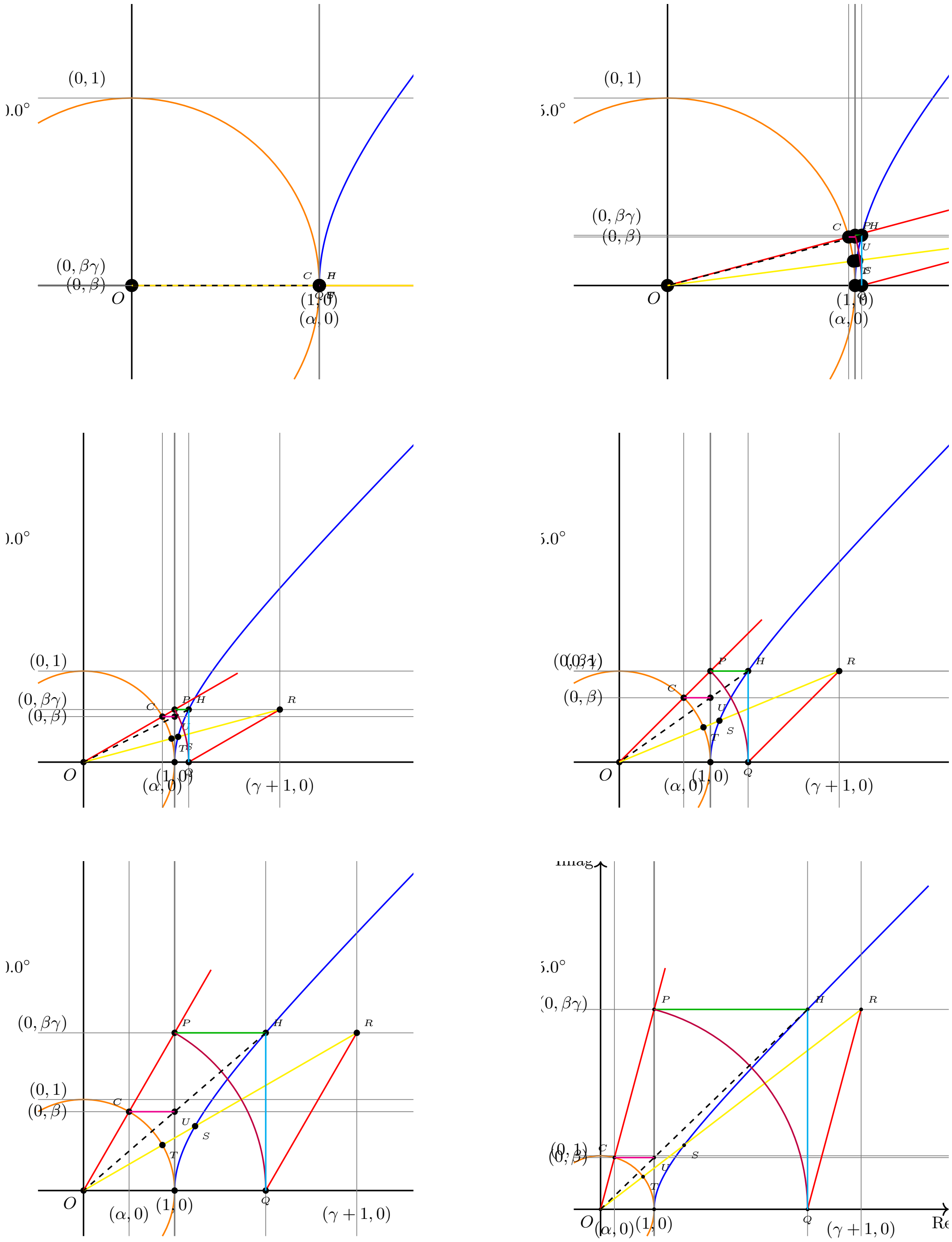

Figure 1: Adaptive Zoom Gudermannian Identities ($\theta = 0°$ to $75°$)

Figure 1: The λ 6-group geometry showing tilt θ and boost η. Points C, P, U, H, Q illustrate the identities (1)–(6).

Comparison of Euclidean Coordinates (tilt angle = 60° for 0.866c)

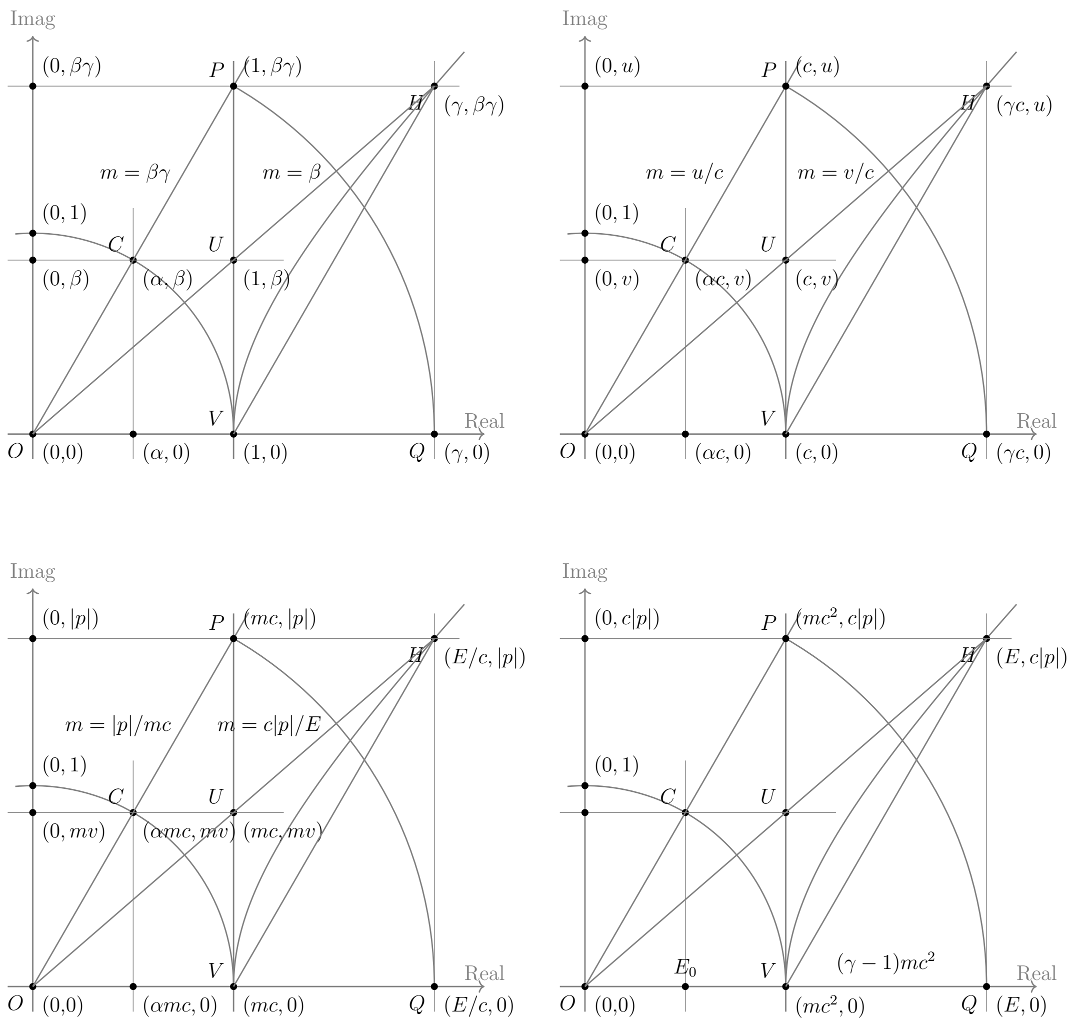

Figure 2: Four-panel comparison: Dimensionless, Velocity, Momentum, Energy — all derived from the same Euclidean geometry.